An introduction to Schramm-Loewner evolution SLE_κ
Course syllabus

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Overview

A hundred years ago, Charles Loewner showed the evolution of maps g_t from the slit disk $\mathbb{D}\backslash\gamma([0,t])$ back to $\mathbb{D}$, where γ is a curve growing into $\mathbb{D}$ from its boundary, satisfy a differential equation which in effect transforms γ into a continuous *driving function* $\lambda(t) = g_t(\gamma(t))$ taking values on $\partial\mathbb{D}$. Loewner's approach played an important role in de Branges' proof of the Bierberbach conjecture in 1985, and received renewed interest following the ground-breaking 2000 work of Oded Schramm, who showed that the random curves generated by using a Brownian driving function run at speed κ give the only-possible conformally-invariant scaling limits of a number of discrete models from statistical physics. These processes, typically normalized to live in the upper half plane $\mathbb{H}$ now instead of Loewner's $\mathbb{D}$, and known as *Schramm-Loewner-Evolutions* SLE_κ, have been intensely studied since and continue to be a topic of active research.

In this course we explore this story, briefly covering the above history but primarily focusing on the post-2000 developments and the beautiful mathematics that emerged following Schramm's work. For the majority of the class we will seek to build the toolbox that combines complex analysis and stochastic calculus to give the foundational results of the theory. Towards the end we will also try to sketch the more "modern" approach following the breakthrough works of Sheffield-Miller connecting SLE to the Gaussian Free Field. There will also be opportunities for student presentations, and flexibility in the content covered to meet students' interests.

Prerequisites

Students should have some background in elementary complex analysis and probability; we will review more advanced results from conformal mapping theory, and also recall key results from stochastic calculus. Lectures will be in English due to the low Mandarin level of the instructor.

Topics covered

These are provisional and open to student requests. They also may not be necessarily covered in exactly this order.

- Intuition and motivation from statistical mechanics models
- Loewner equation and complex analysis background
  - Summary of tools from geometric function theory (Riemann maps, Koebe distortion, Beurling's estimate, invariance of Brownian motion)
  - Hulls, half-plane capacity and Loewner's equation ($\mathbb{H}$ and $\mathbb{D}$)
  - Convergence of Loewner chains and pathologies
- Review of stochastic calculus
- Schramm's theorem: SLE_κ only possible scaling limit
- Existence of SLE trace, Rohde-Schramm theorem, phase transition as κ varies
- Convergence of certain discrete models to SLE_κ
- Properties of trace (Hausdorff dimension, green's function, Holder continuity, reversibility etc)
- Duality; continuity of the trace κ
- Large deviations as $\kappa \to 0^+$
- $\text{SLE}_\kappa(\rho)$ processes and change of coordinates
- SLE_4 as level set of GFF; modern viewpoint: quantum surfaces and quantum zipper

References: courses, lecture notes and textbooks

There are a wealth of outstanding lecture notes and textbooks on SLE. We will primarily follow references (*ii*), (*iii*) and (*vi*) below, and refer to the others as needed for supplementary exposition.

(*i*) Binder, Ilia, *Schramm-Loewner evolution and lattice models.* 2021 online course at the Field's Institute.

(*ii*) Berestycki, Nathanael and J.R. Norris, *Lectures on Schramm-Loewner Evolution.* Online lecture notes, 2016.

(*iii*) Kemppainen, Antti, *Schramm-Loewner Evolution.* Spring Briefs in Mathematical Physics volume 24, 2017.

(*iv*) Lawler, Greg, *Conformally invariant processes in the plane.* AMS Mathematical Surveys and Monographs volume 114, 2008.

(*v*) Lawler, Greg, *Schramm-Loewner evolution.* Online lecture notes, 2007.

(*vi*) Miller, Jason, *Schramm-Loewner evolutions.* Online lecture notes, 2019.

(*vii*) Werner, Wendelin, *Random planar curves and Schramm-Loewner evolutions.* Online lecture notes, 2002.

Supplementary references: complex analysis

(*i*) Duren, Peter. *Univalent functions.* Spring, 1983.

Covers relevant background in conformal mappings and geometric function theory, including Loewner's proof of the 3rd coefficient of the Bieberbach conjecture using Loewner evolution.

(*ii*) Gamelin, Theodore. *Complex analysis.* Springer, 2013.

Easy-to-read reference for basic points in complex analysis.

(*iii*) Lawler, Greg, *Conformally invariant processes in the plane.* AMS Mathematical Surveys and Monographs volume 114, 2008.

Sections on complex analysis cover basic geometric function theory results.

(*iv*) Marshall, Donald. *Complex Analysis.* Cambridge University Press, 2019.

A serious textbook covering all the fundamentals of complex analysis.

Supplementary references: stochastic calculus

(*i*) Kemppainen, Antti, *Schramm-Loewner Evolution.* Spring Briefs in Mathematical Physics volume 24, 2017.

Chapter 2 and online appendices review stochastic calculus and probability theory.

(*ii*) Lawler, Greg, *Conformally invariant processes in the plane.* AMS Mathematical Surveys and Monographs volume 114, 2008.

Has a succinct review section on stochastic calculus.

(*iii*) Lawler, Greg. *Notes on the Bessel Process.* Online textbook, 2019.

(*iv*) Lawler, Greg. *Stochastic Calculus: An Introduction with Applications.* Online textbook, 2014.

Readable notes on Brownian motion, stochastic calculus.

(*v*) Oksendal, Bernt, *Stochastic Differential Equations: An Introduction with Applications.* Springer, 2003.

The first chapters provide a very readable intro to stochastic calculus.