Discrete Markov chains and mixing times
Homework 2

Due 3 月 18 日（周一）at the start of class

1 Textbook problems

- Chapter 2: 2.8
- Chapter 3: 3.1, 3.2
- Chapter 4: 4.1, 4.2, 4.3, 4.4

2 Additional problems

1. Let G be a group and $\mu \in \mathcal{P}(G)$. Suppose that $S := \operatorname{supp}(\mu) := \{ g \in G : \mu(g) > 0 \}$ is a subgroup of G.

   (a) Show that the essential communicating classes of the random walk on G with increment distribution μ are the cosets of S.

   (b) Describe the space of stationary measures for this Markov chain.

2. The distributions π we seek to sample from typically satisfy $\operatorname{supp}(\pi) = \mathcal{X}$ (Ising model, uniform sample, etc), and so the discussion of the Metropolis algorithm in §3.2.1 of Levin-Peres does not explicitly handle the case when $\pi(x)$ vanishes for some states x.

   Let $\Psi \in \mathbb{M}_n^I$ be an irreducible transition matrix, symmetric for simplicity, and suppose we want to modify its associated Markov chain as in the Metropolis algorithm, but are interested in sampling from $\pi \in \mathcal{P}(\mathcal{X})$ for which $\operatorname{supp}(\pi) \subsetneq \mathcal{X}$.

(*a*) What should $a(x,y)$ be for $x \in \mathcal{X}$ such that $\pi(x) = 0$ and $\pi(y) \neq 0$?

(*b*) What should $a(x,y)$ be for $x \in \mathcal{X}$ such that $\pi(x) = 0$ and $\pi(y) = 0$?

(*c*) Show that with your above choices for $a(x,y)$ the resulting chain has stationary distribution π and is reversible under π.

(*d*) Suppose $|\mathcal{X}| \geq 2$. Does your modified Metropolis algorithm "work" for sampling $\pi = \delta_x$, for some $x \in \mathcal{X}$? (Of course, Markov Chain Monte Carlo methods are not necessary for sampling Dirac masses!)

(*e*) By assumption, the original chain under Ψ has a unique essential communicating class. Describe the communicating classes under your version of the Metropolis algorithm, and classify each as essential or inessential.

3. Let P be the transition matrix for the left random walk on a group G with increment distribution μ, and Q the transition matrix for the *right* random walk on G with increment distribution μ. Let $\nu \in \mathcal{P}(G)$. Show that, for any $\epsilon > 0$, we have the equality of mixing times

$$t_{\text{mix}}(P, \nu, \epsilon) = t_{\text{mix}}(\hat{Q}, \hat{\nu}, \epsilon),$$

where $t_{\text{mix}}(R, \rho, \epsilon)$ is the ϵ-mixing time of the Markov chain $(X_t)_{t \geq 0}$ under transition matrix R with initial distribution $X_0 \sim \rho$, and where $\hat{\nu}(g) := \nu(g^{-1})$, as usual.

4. Fix $d \in \mathbb{N}$. Explicitly describe an infinite class of non-constant harmonic functions on the (infinite) lattice $\mathbb{Z}^d$.