Discrete Markov chains and mixing times
Homework 3

Due 4 月 15 日（周一）at the start of class.

1 Textbook problems

- Chapter 5: 5.1, 5.2, 5.4, 5.5
- Chapter 6: 6.2, 6.4, 6.8, 6.10

2 Additional problems

1. Give an example of a random walk on $\mathbb{Z}$ which shows that the bound in Proposition 5.7 is (asymptotically) sharp.

2. Let $G = (V, E)$ be a graph. A *forest* F in G is a subgraph that has no cycles, i.e. a collection $F = \{T_1, \ldots, T_n\}$ of trees T_j in Λ, where T_j and T_k share no vertices if $j \neq k$. The *arboreal gas model* is family of probability measures $\{\mathbb{P}_{\beta,h}\}$ on the collection $\mathcal{X} = \mathcal{F}(G)$ of all forests in G, parametrized by the *inverse temperature* $\beta > 0$ and the *magnetic field* $h \geq 0$. For $F \in \mathcal{F}$,

$$\mathbb{P}_{\beta,h}(F) := \frac{1}{Z(\beta,h)} \beta^{|E(F)|} \prod_{T_j \in F} (1 + h|V(T_j)|),$$

   where $|E(F)|$ is the number of edges in F, and $|V(T_j)|$ is the number of vertices in the tree T_j, and $Z(\beta, h)$ is the usual partition function.

   (a) Describe qualitatively how the measure changes when the parameters β and h vary.

(b) Describe how the Glauber dynamics on $\mathcal{F}$ for $\mathbb{P}_{\beta,h}$. Give explicit transition probabilities. (There are four possible cases.) Explain how the dynamics extends to the entire collection $\tilde{F}$ of all subgraphs (not just forests) of G.

(c) For fixed $\mathbb{P}_{\beta,h}$, construct a grand coupling $\{(X_t^f)_{t\geq 0} : f \in \tilde{\mathcal{F}}\}$.

(d) We would like to imitate the proof strategy for Theorems 5.8 and 5.9 to give bounds on the mixing time for the Glauber dynamics for $\mathbb{P}_{\beta,h}$. Define a natural metric ρ on $\tilde{F}$ and compute $\mathbb{P}(\rho(X_1^f, X_1^g) = 0)$ given that $f, g \in \tilde{\mathcal{F}}$ satisfy $\rho(f, g) = 1$.

(e) Unfortunately we can't finish the argument using the same ideas. What breaks down? (Understanding this helps us appreciate the difficulty of the arboreal gas model in comparison to the q-coloring and hardcore models.)