4.4 Standarizing Distance From Stationarity

(Recall) Thm 4.9: $\max_{x \in \mathcal{X}} \|P^t(x,\cdot) - \pi\|_{TV} \leq C\alpha^t$

(irreducible, aperiodic)

Hence we can define:

$$d(t) \overset{\text{def}}{=} \max_{x \in \mathcal{X}} \|P^t(x,\cdot) - \pi\|_{TV}$$

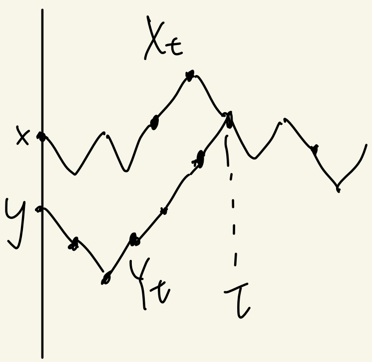
We also introduce:

$$\bar{d}(t) = \max_{\substack{x \in \mathcal{X} \\ y \in \mathcal{X}}} \|P^t(x,\cdot) - P^t(y,\cdot)\|_{TV}$$

(Motivation) $\boxed{\text{coupling}}$ (chapter 5. coupling of 2 chains instead of 2 r.v.)

☆

e.g.



(If $X_s = Y_s$ then $\forall t \geq s$)

$X_t = Y_t$ (*)

Thm 5.4. $\|P^t(x,\cdot) - P^t(y,\cdot)\|_{TV} \leq \mathbb{P}_{xy}(\tau > t)$

(given (*))

convenient to bound $\bar{d}(t)$ instead of $d(t)$.

(2 properties of $d(t)$ & $\bar{d}(t)$)

Lemma 4.10: $d(t) \le \bar{d}(t) \le 2d(t)$ (constant multiple of each other)

Lemma 4.11 $\bar{d}(s+t) \le \bar{d}(s)\,\bar{d}(t)$ (submultiplicative)

Pf of 4.10:

- $\|P^t(x,\cdot) - P^t(y,\cdot)\|_{TV} \le \|P^t(x,\cdot) - \pi\|_{TV} + \|P^t(y,\cdot) - \pi\|_{TV}$

(triangular ineq)

$\max_{x,y\in\mathcal{X}}$

$\Rightarrow \bar{d}(t) \le 2d(t)$

- $\|P^t(x,\cdot) - \pi\|_{TV} = \max_{A\subseteq\mathcal{X}} |P^t(x,A) - \pi(A)|$ (def of TV distance)

$\pi P = \pi$

$$|P^t(x,A) - \pi(A)|$$
$$= \left|\sum_{y\in\mathcal{X}} \pi(y)\left[P^t(x,A) - P^t(y,A)\right]\right|$$
$$\le \sum_{y\in\mathcal{X}} \pi(y)\underbrace{\left|P^t(x,A) - P^t(y,A)\right|}_{\le \bar{d}(t)}$$

$\max_{A\subseteq\mathcal{X}}$

$$\|P^t(x,\cdot) - \pi\|_{TV} \le \bar{d}(t)$$

$\Rightarrow d(t) \le \bar{d}(t)$ ▨

Pf 4.11

- $\|\mu-\nu\|_{TV} = \inf \{ \mathbb{P}(X \neq Y) : (X,Y) \text{ is coupling of } \mu \,\&\, \nu \}$

$\hookrightarrow$ min (inf can be obtained by choosing appropriate coupling)

"optimal"

$$\|P^s(x,\cdot) - P^s(y,\cdot)\|_{TV} = \mathbb{P}(X_s \neq Y_s)$$

(X_s, Y_s) : "optimal coupling"

$\left(\{X_t\}_{t \geq s} \;\&\; \{Y_t\}_{t \geq s} \right)$

- $P^{s+t}(x,w) = \sum_z \mathbb{P}(X_s = z) P^t(z,w)$

$$= \mathbb{E}(P^t(X_s, w))$$

- $P^{s+t}(x,A) - P^{s+t}(y,A)$

$$= \sum_{w \in A} \left(P^{s+t}(x,w) - P^{s+t}(y,w) \right)$$

$$= \sum_{w \in A} \mathbb{E}\left[P^t(X_s, w) - P^t(Y_s, w) \right]$$

$$= \mathbb{E}\left[P^t(X_s, A) - P^t(Y_s, A) \right]$$

$\longrightarrow$ the sum is $\sum_{x,y} \mathbb{P}(X_s = x, Y_s = y) \left[P^t(x,A) - P^t(y,A) \right]$

$\left[P^t(x,A) - P^t(y,A) \right] \leq \bar{d}(t)$

$x \neq y$: $\leq \mathbb{P}(X_s = x, Y_s = y)\,\bar{d}(t)$

$x = y$: no contribution

$$\leq \mathbb{E}\left[\bar{d}(t)\, \mathbb{1}(X_s \neq Y_s) \right]$$

$$= \mathbb{P}(X_s \neq Y_s)\, \bar{d}(t)$$

$\therefore \bar{d}(s+t) \le \bar{d}(s)\, \bar{d}(t)$ ☒

(A simple application)

(Remark 4.12) Another pf of 4.9 using 4.10 & 4.11

Thm 4.9 : $d(t) \le C\alpha^t$

Pf: Suffices to show $\bar{d}(t) \le C\alpha^t$

- Aperiodicity: $\exists s$. s.t. $P^s(x,y) > 0$. $\forall x, y \Rightarrow \bar{d}(s) < 1$
- $\bar{d}(as+b) \le (\bar{d}(s))^a \bar{d}(b)$. $(0 \le b \le s-1,\ a \in \mathbb{N})$

$$\underbrace{\bar{d}(\underbrace{as+b}_{t})}_{} \le \underbrace{\left[(\bar{d}(s))^{\frac{1}{s}} \right]}_{\alpha}{}^{\overbrace{as+b}^{}_{t}} \cdot \underbrace{\left(\bar{d}(s)^{-1} \max_{0 \le b \le s-1} \bar{d}(b) \right)}_{C}$$