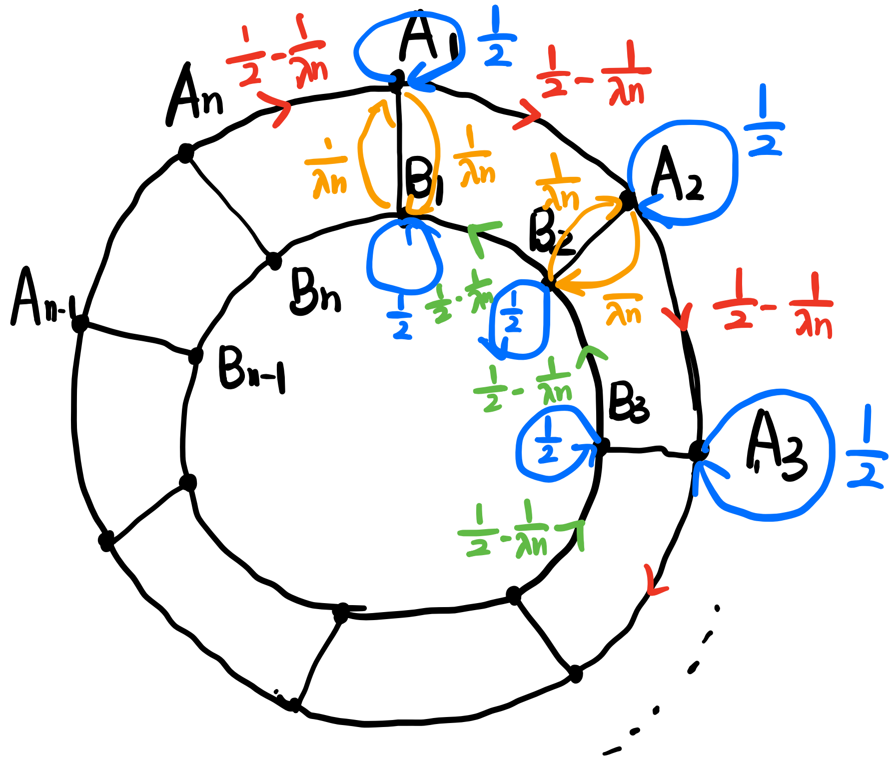
Model:



Thm: $\forall \varepsilon > 0$, $\exists \lambda_0$ sufficiently large $\forall \lambda > \lambda_0$

$$TV\left(X_{2\lambda^2 n},\ \mathrm{unif}\{A_1, \dots, A_n, B_1, \dots, B_n\}\right) \leq \varepsilon$$

Define $\phi(X_t) = \begin{cases} A & X_t \in \{A_1, \cdots, A_n\} \\ B & X_t \in \{B_1, \dots, B_n\} \end{cases}$

$\psi(X_t) = k$ iff $X_t \in \{A_k, B_k\}$

2-step argument

(1) $TV(\phi(X_{\lambda^2 n}),\ \mathrm{unif}\{A, B\}) \leq \frac{\varepsilon}{2}$ for $\lambda > \lambda_0$

(2) $TV(X_{2\lambda^2 n},\ \mathrm{unif}\{A_1, \dots, A_n, B_1, \dots, B_n\}) \leq \varepsilon$ for $\lambda > \lambda_0$

Step (1): suppose $\phi(X_0) = A$

$$N_t := \#\{1 \le i \le t : \phi(X_i) \ne \phi(X_{i-1})\}$$

$$N_{\lambda^2 n} \stackrel{d}{=} \mathrm{Bin}(\lambda^2 n, \tfrac{1}{\lambda n}) \approx \mathrm{Pois}(\lambda)$$

$$\mathbb{P}(\phi(X_{\lambda^2 n}) = A) \approx \mathbb{P}(\mathrm{Pois}(\lambda) \text{ is even}) = \tfrac{1}{2} - \tfrac{\varepsilon}{2}$$

for λ sufficiently large $\blacksquare$

Step 2: based on Step (1), suffice to show

$$TV(\psi(X_{\lambda^2 n}), \mathrm{unif}\{1, \dots, n\}) \le \frac{\varepsilon}{2} \quad \text{for suff. large } \lambda$$

suffices to consider the driftless chain. WLOG $X_0 = A_0$

$$\tau_1 := \inf_{t>0} \{\phi(X_t) = B\}, \quad \tau_2 := \inf_{t>\tau_1} \{\phi(X_t) = A\}$$

hope to show
$$\begin{cases} \tau_1 + \tau_2 \le \lambda^2 n \quad \text{with prob.} \ge 1 - \frac{\varepsilon}{10} \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \text{(easy!)} \\ \psi(X_{\tau_1+\tau_2}) \text{ is close to uniform } \{1, \dots, n\} \end{cases}$$

$$\psi(X_{\tau_1+\tau_2}) = \tau_1 - \tau_2 \qquad \tau_1, \tau_2 \stackrel{i.i.d}{\sim} \mathrm{Geo}(\tfrac{1}{\lambda n})$$

suffice to show $\left[\mathrm{Geo}\left(\frac{1}{\lambda n}\right) - \mathrm{Geo}\left(\frac{1}{\lambda n}\right)\right] \bmod n$ is close to $\mathrm{unif}\{1, \dots, n\}$

"continues version" $Z_1, Z_2 \overset{d}{=} \mathrm{Exp}\left(\frac{1}{\lambda}\right)$

- $\mathbb{P}(Z_1 \geq t) = \mathbb{P}(Z_2 \geq t) = e^{-\frac{t}{\lambda}}$

$\{Z_1 - Z_2\}$ non-integer part of $Z_1 - Z_2$

Then $P_{\{Z_1 - Z_2\}}(t) \longrightarrow \mathbb{1}_{(0,1)}(t)$ as $\lambda \to \infty$

$P_{Z_1 - Z_2}(t) = \frac{1}{2} e^{-\frac{|t|}{\lambda}}, \quad t \in \mathbb{R}$

$$P_{\{Z_1 - Z_2\}}(t) = \sum_{n \in \mathbb{Z}} P_{Z_1 - Z_2}(n+t)$$

$$= \frac{1}{2}\left(e^{\frac{t}{\lambda}} + e^{-\frac{t}{\lambda}}\right) \cdot \frac{1}{1 - e^{-\frac{1}{\lambda}}} \xrightarrow{\lambda \to \infty} 1, \quad t \in (0,1)$$

$\blacksquare$