3.3 Glauber Dynamics

MC. state spaces: S^V, where V is the vertex set of a graph and S is a finite set.

the elements of S^V called configurations

- a labeling of vertices with elements of S.

prob. measure π. the Glauber dynamics for π

- a MC. with stationary distribution π.
- also called Gibbs sampler

3.3.1 Two examples.

Example 1. proper q-coloring of a graph $G=(V,E)$ $\longrightarrow$ $x \in \{1,\dots,q\}^V$

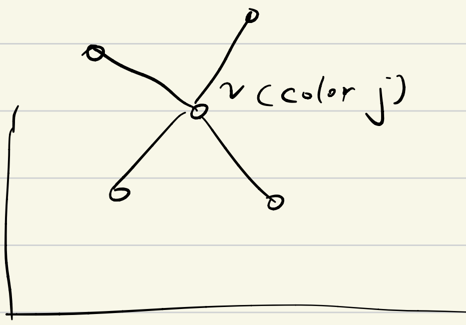
- $x(v) \neq x(w)$, if $v \sim w$.

Def. a color j is allowable at v if $j \notin \{x(w) : w \sim v\}$.

Glauber dynamics for proper q-colorings

Step 1. select a vertex $v \in V$ at random (uniformly)

Step 2. Select a color j uniformly at random from the allowable colors at v and re-color vertex v with color j.



Claim: uniform stationary distribution

in fact, reversible. (Check DBE)

- x, y agree everywhere except vertex v.

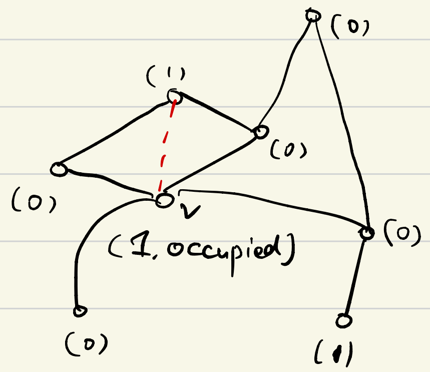
$$P(x,y) = \underbrace{|V|^{-1}}_{\text{Step 1}} \cdot \underbrace{|A_v(x)|^{-1}}_{\substack{\text{Step 2} \\ \text{(allowable colors)}}}$$

$\Rightarrow$ $P(x,y) = P(y,x)$ (since $|A_v(x)| = |A_v(y)|$)

Note. Step 2 : Sample from π conditioned on the set of colorings agreeing with x at all vertices except v.

Example 2. hardcore configuration

Def. a placement of particles on V so that each vertex is occupied by at most one particle and no two particles are adjacent.



$x \in \{0,1\}^V$: $x(v)\,x(w) = 0$, if $v \sim w$.

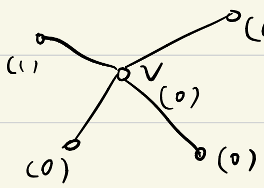
$\left(\begin{array}{ll} x(v) = 1, & \text{occupied} \\ x(v) = 0, & \text{vacant} \end{array} \right)$

Glauber dynamics:

Step 1 : choose a vertex v uniformly at random

Step 2 : if any neighbour of v is occupied, v is left unoccupied.

(If there exists a neighbour)



Remark 3.4 : $\boxed{\text{Couple}}$ multiple copies of this chain

if no adjacent vertex is occupied, v is occupied with prob. $1/2$ and vacant with prob. $1/2$.

Equivalently, Step 2 : If no neighbor of v is occupied, then with prob. $1/2$, flip the status of v.

- reversible w.r.t. the uniform distribution

(similar to the coloring chain)

$P(x,y) = |V|^{-1} \cdot \frac{1}{2}$, $P(y,x) = |V|^{-1} \cdot \frac{1}{2}$

3.3.2 General definition

V and S finite sets

$\mathcal{X} \subseteq S^V$. (state space)

prob. measure π supported on $\mathcal{X}$

The (single-site) Glauber dynamics for π is a reversible MC with state space $\mathcal{X}$, stationary distribution π.

trans. prob.

Step 1: choose v uniformly at random from V.

Step 2: Sample a new state from π conditioned on the set of states equal to x at all vertices different from v.

$\uparrow$ the original state

$(\pi^{x,v}(y))$

For $x \in \mathcal{X}$ and $v \in V$, let

$$\mathcal{X}(x,v) = \{y \in \mathcal{X} : y(w) = x(w) \text{ for all } w \neq v\}$$

$$\pi^{x,v}(y) = \pi(y \mid \mathcal{X}(x,v)) = \begin{cases} \dfrac{\pi(y)}{\pi(\mathcal{X}(x,v))}, & \text{if } y \in \mathcal{X}(x,v) \\ 0, & \text{if } y \notin \mathcal{X}(x,v) \end{cases}$$

$\uparrow$ conditional prob.

3.3.4 Hardcore model with fugacity

$G = (V, E)$, $\mathcal{X}$ is the set of hardcore configurations on G.

Hardcore model with fugacity λ is the prob. distribution π on hardcore config. $x \in \{0,1\}^V$ defined by

$$\pi(x) = \begin{cases} \dfrac{\lambda^{\sum_{v \in V} x(v)}}{Z(\lambda)}, & \text{if } x \text{ is a hardcore config.} \\ 0, & \text{otherwise} \end{cases}$$

$\propto \lambda^{\sum_{v \in V} x(v)}$

$\uparrow$ parameter

(Note $\lambda = 1$) standard version

Glauber. Step 1: Choose w uniformly at random

Step 2: $X_{t+1}(w) = \begin{cases} 1, & \text{with prob. } \frac{\lambda}{1+\lambda} \\ 0, & \text{with prob. } \frac{1}{1+\lambda} \end{cases}$, if no neighbour of w is occupied.

$$\frac{P(X_{t+1}(w) = 1 \mid \cdots)}{P(X_{t+1}(w) = 0 \mid \cdots)} = \lambda$$

intuition: $\lambda > 1$, tendency to more particles