Spectral gap and mixing

Assume: P aperiodic.

Recall: $\forall x, y.\ P^t(x,y) \to \pi(y)\ (t \to \infty)$

mixing time:

$$t_{mix}(\varepsilon) := \min\{t \geq 0 : d(t) \leq \varepsilon\}$$

$$d(t) = \max_{x \in V} \|P^t(x,\cdot) - \pi(\cdot)\|_{TV}$$

a different notion of distance.

Def 5.2.9 Separation distance.

$$s_x(t) := \max_{y \in V} \left[1 - \frac{p^t(x,y)}{\pi(y)}\right].$$

$$s(t) = \max_{x \in V} s_x(t).$$

Recall Lemma 5.2.10. compare s-d and TV-d.

$$d(t) \leq s(t)$$

pf: $\|P^t(x,\cdot) - \pi(\cdot)\|_{TV}$.

$$= \sum_{y: p^t(x,y) < \pi(y)} [\pi(y) - p^t(x,y)]$$

$$= \sum_{y: p^t(x,y) < \pi(y)} \pi(y) \left[1 - \frac{p^t(x,y)}{\pi(y)}\right]$$

$$\leq s_x(t).$$

$\square$

Recall spectral decomposition of P^t.

Thm 5.2.7 $\quad P$ irreducible, reversible

eigenvalue $1=\lambda_1 \geq \cdots \geq \lambda_n$

eigenfunction $f_1, \cdots, f_n$

then $\dfrac{P^t(x,y)}{\pi(y)} = 1 + \sum_{j=2}^{n} f_j(x) f_j(y) \lambda_j^t$

☆: speed of $P^t(x,y) \to \pi(y)$

is dominated by the largest eigenvalue

of P not equal to 1

Def 5.2.11 (spectral gap)

absolute spectral gap $\gamma_* := 1 - \underset{|\lambda_2| \vee |\lambda_n|}{\underset{\shortparallel}{\lambda_*}}$

spectral gap: $\gamma := 1 - \lambda_2$

Rmk: ① from Lemma 5.2.3 − 5.2.4 $(\lambda_2 < 1)$ $(\lambda_n > -1)$ — exercise 5.9.

$\gamma_* > 0$ if P $\begin{cases} \text{irreducible} \\ \text{aperiodic} \end{cases}$

② lazy version $\frac{1}{2}(P+I)$

eigenvalue $\frac{1}{2}(\lambda_j + 1) \geq 0$

$\Rightarrow \gamma_* = \gamma$.

Def: 5.2.12 Relaxation time.

$t_{rel} := \gamma_*^{-1}$.

Example 5.2.13 two-state chain

$$V=\{0,1\} \quad P=\begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} \quad \alpha, \beta\in(0,1)$$

$$\pi=\left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right).$$

$$f_1=1. \quad \lambda_1=1 \qquad f_2=\left(\sqrt{\frac{\alpha}{\beta}}, -\sqrt{\frac{\beta}{\alpha}}\right) \lambda_2=1-\alpha-\beta.$$

$$P^t=\begin{pmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{pmatrix}+(1-\alpha-\beta)^t\begin{pmatrix} \frac{\alpha}{\alpha+\beta} & -\frac{\alpha}{\alpha+\beta} \\ -\frac{\beta}{\alpha+\beta} & \frac{\beta}{\alpha+\beta} \end{pmatrix}$$

For $d(t)$: $\quad d(t)=\max_x \frac{1}{2}\sum_y |P^t(x,y)-\pi(y)|$

$$=\frac{\beta}{\alpha+\beta}|1-\alpha-\beta|^t.$$

$$t_{mix}(\varepsilon)=\left\lceil \frac{\log\varepsilon^{-1}-\log\left(\frac{\alpha+\beta}{\beta}\right)}{\log|1-\alpha-\beta|^{-1}} \right\rceil$$

For t_{rel}: $\quad \lambda_2=1-\alpha-\beta.$

$$\gamma_*=\begin{cases} \alpha+\beta: & \alpha+\beta\le 1 \\ 2-\alpha-\beta. & \alpha+\beta>1. \end{cases}$$

$$t_{rel}=\begin{cases} \frac{1}{\alpha+\beta} & \alpha+\beta\le 1 \\ \frac{1}{2-\alpha-\beta} & \alpha+\beta>1 \end{cases}$$

intuition $d(t) \simeq (1-\gamma^*)^t$

Thm: 5.2.14 P reversible, irreducible, aperiodic

$\pi(x) > 0 \quad \forall x$.

$\forall \varepsilon > 0 \qquad (t_{rel} - 1) \log\left(\frac{1}{2\varepsilon}\right) \leq t_{mix}(\varepsilon) \leq \log\left(\frac{1}{\varepsilon \pi_{min}}\right)$ t_{rel}.

Pf: upper bound: find t s.t. $s(t) \leq \varepsilon$.

$$\left| \frac{P^t(x,y)}{\pi(y)} - 1 \right| \leq |\lambda_*|^t \sum_{j=2}^{n} |f_j(x) f_j(y)|$$

Lemma 5.2.6: $\sum_{j=1}^{n} f_j(x) f_j(y) = \frac{\delta_x(y)}{\pi(x)}$

$$\text{RHS} \leq \frac{|\lambda_*|^t}{\sqrt{\pi(x)\pi(y)}} = \frac{(1-\gamma_*)^t}{\pi_{min}} \leq \frac{e^{-\gamma_* t}}{\pi_{min}}$$

$$\Rightarrow s(t) \leq \frac{e^{-t/t_{rel}}}{\pi_{min}}$$

$$\Rightarrow t_{mix}(\varepsilon) \leq \log\left(\frac{1}{\varepsilon \pi_{min}}\right) \cdot t_{rel}.$$

lower bound: z s.t. $|f_*(z)| = \|f_*\|_\infty$

Lemma 5.2.5 $\quad \pi f_* = 0$

$$\lambda_*^t |f_*(z)| = |P^t f_*(z)| = \left| \sum_y (P^t(z,y) - \pi(y)) f_*(y) \right|$$

$$\leq \|f_*\|_\infty \cdot 2 d(t)$$

$\Rightarrow d(t) \geq \frac{1}{2}\lambda_*^t.$

when $t = t_{mix}(\varepsilon)$. $\quad \varepsilon \geq \frac{1}{2}\lambda_*^{t_{mix}(\varepsilon)}$

$$\Rightarrow \quad t_{mix} \geq \frac{\log\left(\frac{1}{2\varepsilon}\right)}{\log\left(\frac{1}{\lambda_*}\right)} = \frac{\log\left(\frac{1}{2\varepsilon}\right)}{\log\left(\frac{1}{1-t_{rel}^{-1}}\right)}$$

$$\geq \frac{\log\left(\frac{1}{2\varepsilon}\right)}{\frac{t_{rel}^{-1}}{1-t_{rel}^{-1}}} = (t_{rel}-1)\log\left(\frac{1}{2\varepsilon}\right).$$